Problem Set 1

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Due: 9/24, 11:59pm.
Please typeset your solutions in LaTeX.

Problem 1 (SoS proofs beyond eigenvalues)**.** We saw in lecture that if $M \in \mathbb{R}^{n \times n}$ is a symmetric matrix with maximum eigenvalue λ_{max}, then there is always a degree-2 SoS proof that $x^\top M x \le \lambda_{max} \cdot \|x\|^2$ – that is, the polynomial $\lambda_{max} \cdot \|x\|^2 - x^\top M x$ is a sum of squares.

1. Show that this bound is tight, in the sense that if c is such that $c\|x\|^2 - x^\top M x$ is a sum of squares, then $c \ge \lambda_{max}$.
2. Construct a symmetric matrix M such that there exists $c < \lambda_{max}(M)$ and linear functions $f_1, \ldots, f_m$ such that $c \cdot \|x\|^2 - x^\top M x = \sum_{i \le m} f_i(x)^2$ for every $x \in \{\pm 1\}^n$. This shows that the flexibility in a quadratic proof to use a sum of squares polynomial which is equal to $c\|x\|^2 - x^\top M x$ only for certain xs (namely, $x \in \{\pm 1\}^n$) makes the definition more powerful.

Problem 2 (Max cut in almost-bipartite graphs)**.** Show that there is a polynomial-time algorithm with the following guarantee: given a graph $G = (V, E)$ such that there is a cut which cuts $(1 - \varepsilon)|E|$ edges, the algorithm outputs a cut which cuts $(1 - \tilde{O}(\sqrt{\varepsilon}))|E|$ edges. ($\tilde{O}$ can hide factors of $\log(1/\varepsilon)$, though this is not strictly necessary.)

You may use the following basic anticoncentration fact for Gaussians: if $Z \sim N(0, 1)$, then $Pr(|Z| \le \delta) = O(\delta)$.

Problem 3 (Cauchy-Schwarz for Pseudoexpectations)**.** An important fact about any probability distribution μ is that for any real-valued f and g, $\mathbb{E}_{x \sim \mu} f(x) g(x) \le \sqrt{\mathbb{E} f(x)^2} \cdot \sqrt{\mathbb{E} g(x)^2}$. Show that if $\tilde{\mathbb{E}}$ is a (degree 2) pseudoexpectation and f, g are linear functions, one has $\tilde{\mathbb{E}} f(x) g(x) \le \sqrt{\tilde{\mathbb{E}} f(x)^2} \cdot \sqrt{\tilde{\mathbb{E}} g(x)^2}$.

Problem 4 (Max cut on the triangle)**.** The three-edge triangle graph has a max-cut value of 2. We saw in class that there is a quadratic proof that the maximum cut is at most 2.9. Is there a quadratic proof that the max-cut value is at most 2.00001? (Prove the correctness of your answer.)