Problem Set 2

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Due: 10/8, 11:59pm.
Please typeset your solutions in LaTeX.

Problem 1 (On $\vDash$, borrowed from Aaron Potechin)**.** Consider the following polynomial equation in 3 variables, x, y, z.

$$(x^2+1)y = z^2.$$

Because it implies $y = \frac{z^2}{x^2+1}$, any solution (x, y, z) to the above must have $y \geq 0$. We will see if sum-of-squares can capture this reasoning.

1. Construct a degree 4 pseudoexpectation $\widetilde{\mathbb{E}}$ in variables x, y, z such that $\widetilde{\mathbb{E}} \vDash (x^2+1)y = z^2$ but $\widetilde{\mathbb{E}}\, y < 0$. (Computer-aided proofs are allowed.)

   By $\widetilde{\mathbb{E}} \vDash (x^2+1)y = z^2$, we mean that for any polynomial p of degree at most 1 in x, y, z, $\widetilde{\mathbb{E}}\, p(x,y,z)(x^2+1)y = \widetilde{\mathbb{E}}\, p(x,y,z)z^2$.

2. Despite the above, show that there exists a sum-of-squares refutation to the following system of polynomial inequalities, for any $c > 0$: $\{(x^2+1)y = z^2, y \leq -c\}$.

Problem 2. Suppose $\widetilde{\mathbb{E}}$ is a pseudoexpectation of degree d, with d even, and $\widetilde{\mathbb{E}} \vDash p \leq 0, p \geq 0$ for some polynomial p. (Informally, we have been writing $\widetilde{\mathbb{E}} \vDash p = 0$.) Show that if p has even degree, for every q such that the degree of pq is at most d, we have $\widetilde{\mathbb{E}}\, pq = 0$. Similarly, show that if p has odd degree, for every q such that the degree of pq is at most $d-1$, we have $\widetilde{\mathbb{E}}\, pq = 0$.

Problem 3. In class, we saw how Gaussian rounding and global correlation could be used to approximate the max-cut of a graph. In this exercise, we will see how similar ideas can be used for *max-bisection*. Let $G = (V, E)$ be a graph with $|V| = n$ even. The goal in the max-bisection problem is to determine

$$\mathsf{OPT} = \max_{\substack{S \subseteq V \\ |S| = n/2}} E(S, \overline{S}),$$

where $E(S, \overline{S})$ is the size of the cut corresponding to S, that is, the number of edges between S and $\overline{S}$. The goal in this exercise will be to prove the following theorem.

Theorem. *Let G be a regular graph with max-bisection value at least $(1-\varepsilon)|E|$. There exists an algorithm running in time $n^{(1/\varepsilon)^{O(1)}}$ that outputs a bisection cutting $(1 - O(\sqrt{\varepsilon}))|E|$ edges.*

Let $\widetilde{\mathbb{E}}$ be a pseudodistribution over $\{\pm 1\}^n$ such that

$$\widetilde{\mathbb{E}} \vDash \left\{ \frac{1}{4} \sum_{ij \in E} (y_i - y_j)^2 \geq (1-\kappa)|E|, \sum y_i = 0 \right\}.$$

1. Suppose we apply Gaussian rounding to $\widetilde{\mathbb{E}}$ to produce a random vector $z \in \{\pm 1\}^n$. Show that if the global information of $\widetilde{\mathbb{E}}$ is at most δ, then $\mathbf{Var}\left(\sum z_i\right) \leq \delta^{\Omega(1)} \cdot n^2$.

2. For $\delta > 0$, explain how to round $\widetilde{\mathbb{E}}$ of degree $\mathrm{poly}(1/\delta)$ sufficiently large to a distribution z over $\{\pm 1\}^n$ such that
$$\frac{1}{4}\mathbb{E}\sum_{ij \in E}(z_i - z_j)^2 \geq (1 - O(\sqrt{\kappa}))|E|$$
and
$$\mathbf{Var}\left(\sum z_i\right) \leq \delta n^2.$$

3. Using the above, design a (randomized) algorithm running in time $n^{(1/\varepsilon)^{O(1)}}$ that outputs $z \in \{\pm 1\}^n$ such that $\sum z_i = 0$ and
$$\frac{1}{4}\sum_{ij \in E}(z_i - z_j)^2 \geq (1 - O(\sqrt{\varepsilon}))|E|.$$
Conclude that you have proved the theorem.

Bonus Problem 4 (Integrality gaps for max-cut, borrowed from Pravesh Kothari)**.** Let C_n be the cycle graph on vertex set $[n]$ with edge set E. Further suppose that n is odd. The size of the max-cut in C_n is $n-1$. Recall from your solution to Problem 2 of the first problem set that this implies that for any degree 2 pseudoexpectation $\widetilde{\mathbb{E}}$ on $\{\pm 1\}^n$, $\widetilde{\mathbb{E}}\left[\frac{1}{4}\sum_{ij \in E}(x_i - x_j)^2\right] \leq \left(1 - O\left(\frac{1}{n^2}\right)\right)n$. We will start by seeing that this is tight for degree 2 pseudoexpectations.

Let L the Laplacian of C_n defined by $L_{ii} = 2$ for each i, and L_{ij} is -1 if ij is an edge and 0 otherwise. Observe that for $x \in \{\pm 1\}^n$, the size of the cut associated to x is equal to $\frac{1}{4} \cdot x^\top L x$.

For each $0 \leq k \leq n/2$, let x_k, y_k be vectors with coordinates $(x_k)_i = \cos(2\pi i k/n)$ and $(y_k)_i = \sin(2\pi i k/n)$.

1. Prove that x_k and y_k are eigenvectors of L with eigenvalues $2 - 2\cos(2\pi k/n)$.

2. Prove that the diagonal entries of the matrix $M_k = x_k x_k^\top + y_k y_k^\top$ are 1.

3. Prove that there is a degree 2 pseudoexpectation $\widetilde{\mathbb{E}}_k$ on $\{\pm 1\}^n$ with $\widetilde{\mathbb{E}}\, x = 0$ and $\widetilde{\mathbb{E}}\, x x^\top = M_k$. Using this, prove that for $k = \frac{n-1}{2}$, $\widetilde{\mathbb{E}}\left[\frac{1}{4}\sum_{ij \in E}(x_i - x_j)^2\right] \geq \left(1 - O\left(\frac{1}{n^2}\right)\right)n$.

Next, we will see that degree 6 pseudoexpectations do not face such barriers (for the cycle graph).

4. Prove that for degree 6 pseudoexpectations $\widetilde{\mathbb{E}}$ over $\{\pm 1\}^n$, the squared triangle inequality holds: $\widetilde{\mathbb{E}}(x_i - x_j)^2 \leq \widetilde{\mathbb{E}}(x_i - x_k)^2 + \widetilde{\mathbb{E}}(x_k - x_j)^2$. For a harder exercise, prove this for degree 4 pseudoexpectations.

5. Prove that for any degree 6 pseudoexpectation $\widetilde{\mathbb{E}}$, $\widetilde{\mathbb{E}}\left[\frac{1}{4}\sum_{ij \in E}(x_i - x_j)^2\right] \leq n - 1$.